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ON THE ALGORITHMIC COMPLEXITY OF COLORING SIMPLE HYPERGRAPHS AND STEINER TRIPLE SYSTEMS

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Dedicated to Paul Erdős on his seventieth birthday

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In this paper we establish that deciding t-colorability for a simple k-graph when $t \geqq 3$, $k \geqq 3$ is NP-complete. Next, we establish that if there is a polynomial time algorithm for finding the chromatic number of a Steiner Triple system then there exists a polynomial time "approximation" algorithm for the chromatic number of simple 3-graphs. Finally, we show that the existence of such an approximation algorithm would imply that P=NP.

1. Introduction

A k-uniform hypergraph of order v is a v-set, V, along with a collection, B, of k-subsets of V. A *simple k-graph* then is a k-uniform hypergraph (V, B) such that for any two k-subsets (or edges), $e, e' \in B$, $|e \cap e'| \leqq 1$. Simple k-graphs can also be described as k-uniform hypergraphs without 2-cycles. Alternately, a simple k-graph can be thought of as a partial Steiner system or block design.

A *Steiner system* $S(2, k, v)$ is a v-set P along with a collection, B, of k-subsets of P such that every pair of elements of P is contained in exactly one k-subset (or block) of B. A *partial* $S(2, k, v)$ is similar except that now every pair of elements is contained in at most one block of B. Block designs and simple k-graphs have had little influence on one another in the past. In this paper, we consider a number of questions involving the algorithms complexity of coloring Steiner systems and simple k-graphs.

A *t-coloring* of a simple k-graph is a partition of the vertices V into t color classes so that no k-subset of B is contained in any one color class. The chromatic number, $\chi(S)$, of a simple k-graphs, S, is the smallest integer t such that there exists a t-coloring of S but no $t-1$-coloring. These definitions naturally apply to block designs as well.

Coloring problems for hypergraphs are hard in general. Even deciding whether a hypergraph (P, B) is 2-colorable is NP-complete when edges $e \in B$ have cardinality $|e| > 2$ (Lovász [13]). Restricting the class of hypergraphs to simple k-graph or even k-graphs without cycles of length r, $r \leqq p$, for given $p > 2$, does not improve matters: we will establish that deciding whether such a k-graph is t-colorable, for $t \geqq 3$, is still NP-complete.

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A topic of great interest in block designs involves the finite embeddability of partial designs. In particular, it has been established, for each k, that a partial Steiner $S(2, k, v)$ can be embedded in a Steiner system $S(2, k, n)$ for n sufficiently large. (C. Treash [16], R. Quackenbush [15], B. Ganter [5]—see C. C. Lindner's suıvey article [12]). A question related to finite embeddability, is whether deciding t-colorability of Steiner systems $S(2, k, v)$ (as a sub-class of simple k-graphs) is also NP-complete.

In a previous paper [1], it was established that although 2-colorability of partial Steiner quadruple systems (i.e. 4-graphs) is NP-complete, 2-colorability for Steiner quadruple systems $S(3, 4, n)$ is decidable in polynomial time. Of course even though every partial Steiner quadruple system can be finitely embedded (B. Ganter [6], cf. Lindner [12]), the size and complexity of the embedding is very significant. If we consider only simple 3-graphs and Steiner triple systems $S(2, 3, v)$, then the embedding problem is tractable. Using the work of C. C. Lindner [11] and A. Cruse [2], we can derive a small efficiently computable embedding algorithm. Thus the question of t-colorability of Steiner triple systems become particularly interesting. In this paper we establish that determining the chromatic number of a Steiner triple system is NP-hard, deciding 14-colorability of a Steiner triple system is NP-complete, and determining t-colorability of a simple k-graph is NP-complete $t \geqq 2$.

In a previously published paper, the authors, in conjunction with C. J. Colbourn and M. J. Colbourn [17], established that coloring partial triple systems is NP-complete and moreover coloring blocks designs is NP-complete as well. A *block design* (v, k, λ) is a set of v points, P, and a collection, B, of k-element subsets of P such that every pair of elements in P is contained in exactly λ blocks (or subsets) of B. The result cited above was in fact established for block designs with $k=3$ and $\lambda > 1$ (in fact λ is $O(v)$). These results are weaker than those contained in this paper. However, the arguments [17] are correspondingly shorter as well.

2. Simple k-graphs

In order to establish that the problem of t-coloring simple k-graphs is NP-complete, we reduce a well-known NP-complete problem—the problem of t-coloring graphs—to the t-coloring of simple k-graphs. First, we must establish the following lemmas.

Lemma 2.1. *For each $t \geqq 2$, there exists a t-chromatic simple k-graph for which any proper t-coloring assigns the same color to a particular (fixed) k-subset.*

Proof. There exists $(t+1)$-chromatic simple k-graphs for all $t \geqq 2$ ([3], [4], [14]). Suppose P is a $(t+1)$-chromatic simple k-graph. An edge (k-subset) of P is critical if deleting it lowers the chromatic number of P. Let us assume that e is a critical edge of P and that P^* is the simple k-graph P with e removed. Then P^* is t-chromatic and moreover any t-coloring of P^* must assign the same color to all k elements of the deleted edge — otherwise this would be a t-coloring of P as well, which is impossible. ∎

Lemma 2.2. *For each $t \geqq 2$, there exists a t-chromatic simple k-graph P such that for some (fixed) pair of vertices $\{x, x'\}$ any t-coloring of P assigns different colors to x and x'.*

Proof. Let P be a t-chromatic simple k-graph such that any t-coloring of P assigns the same color to the k-subset $\{x, y_1, y_2, \ldots, y_{k-1}\}$. Such a simple k-graph will exist by our previous lemma. If $P=(V, B)$, choose some element $x' \notin V$ and add the edge $\{x', y_1, y_2, \ldots, y_{k-1}\}$ to the collection B to form a new (simple) k-graph P'. Clearly any t-coloring of P' must color x and x' differently. Note—we can assume that $\{\{x', y_1, \ldots, y_{k-1}\}\} \cup B$ is simple from the proof of Lemma 2.1. ∎

Let $P_t(Q \cup \{x, y\}, B)$ be a t-chromatic simple k-graph with distinguished points x, y where any t-coloring of P_t colors x and y differently. Let $G=(V, E)$ be any graph. For each edge $e \in E$, $e=[u, v]$, construct a copy of P_t on the set $Q \times \{e\} \cup \cup \{u, v\}$ so that u and v are identified with x and y respectively. Denote this simple k-graph by $P^e=[Q^e, B^e]$ and define $F(G)=[Q^*, B^*]$ where $Q^* = \bigcup_{e \in E} Q^e$ and $B^* = \bigcup_{e \in E} B^e$ for $G=(V, E)$. Clearly $F(G)$ is a simple k-graph. In fact if P (the k-graph from which P_t was constructed, Lemma 2.1) did not contain a cycle of length r, $r \leqq p$, for given constant p, then $F(G)$ will contain no cycles of length less than p. Moreover if $\chi(G) \leqq t$ then $\chi(F(G))=t$ and if $\chi(G)>t$ then $\chi(F(G))>t$. (Actually we can insure that if $\chi(G)>t$ then $\chi(F(G))=t+1$ by careful choice of P_t). Since the sizes of $F(G)$ and G are polynomially related (i.e., if G has n vertices and m edges then $F(G)$ has $c_1 m$ vertices and $c_2 m$ edges), we conclude that the existence of a polynomial time algorithm for deciding t-colorability of simple k-graphs implies that there exists a polynomial-time algorithm for deciding t-colorability of graphs. In conclusion we have the following theorem:

Theorem 2.4. *Deciding whether a simple k-graph is t-colorable, is NP-complete for any $t \geqq 3$.* ∎

Corollary 2.5. *For any $p \geqq 2$, deciding whether a k-graph without cycles of length less than p is t-colorable is NP-complete.*

Proof. As was observed in the proof of Lemma 2.2, the same construction applies to these k-graphs as well. ∎

3. Embedding of Partial Triple Systems

C. Treash [16] proved that every partial triple system can be embedded into a finite Steiner triple system. Subsequently, C. C. Lindner proved that any partial triple system of order n can be embedded into a Steiner triple system of order $6n+3$. We require a slightly 'twisted' version of Lindner's construction (cf. Lindner [11], [12]).

Let $P=[V, B]$ be a partial triple system of order n. Define a (partial) quasigroup operation, $*$, by $x*y=z$ if and only if $\{x, y, z\}$ is a triple of B and define $x*x=x$ for all $x \in V$. (A quasigroup is a binary operation which is cancellative.) A. Cruse [2] proved that any (partial) idempotent commutative quasigroup of order n can be embedded into an idempotent commutative quasigroup of order $2n+1$. Let $(W, *)$ be such a quasigroup and let $W_i=\{x_i | x \in W\}$. Define a Steiner triple system $S=[Q, B^*]$ where $Q=W_1 \cup W_2 \cup W_3$ as follows:

(a) for each triple $\{x, y, z\}$ from the partial triple system $P=[V, B]$ include the following blocks in B^*:

$$\{x_1, y_1, z_1\} \quad \{x_1, y_2, z_2\} \quad \{x_1, y_3, z_3\}$$

$$\{x_2, y_2, z_3\} \quad \{x_2, y_3, z_1\} \quad \{x_3, y_2, z_1\}$$

$$\{x_3, y_3, z_2\} \quad \{x_3, y_1, z_3\} \quad \{x_2, y_1, z_2\}.$$

(b) For each pair $\{x, y\} \subseteq W$ where $\{x, y\}$ is not contained in any triple of B, include the following blocks in B^*.

$$\{x_1, y_1, z_2\}, \quad \{x_2, y_2, z_3\}, \quad \{x_3, y_3, z_1\}, \quad \text{where} \quad x * y = z.$$

(c) For each $x \in W$, $\{x_1, x_2, x_3\}$ is included in B^*.

Theorem 3.1. [11] *$S=[Q, B^*]$ is a Steiner triple system of order $6n+3$ containing the partial triple system $P=[V, B]$.* ▮

The next result follows almost immediately from the above construction.

Lemma 3.2. *Let P be a partial triple system of order n and assume that P has been embedded in S, a Steiner triple system, by the above procedure. Then,*

$$\chi(P) \leqq \chi(S) \leqq \chi(P)+2.$$

Proof. Obviously $\chi(P) \leqq \chi(S)$. Assume $\chi(P)=k$ and $C_1, C_2, \dots, C_k$ is a k-coloring of P where $C_1 \cup C_2 \cup \dots \cup C_k = V_1$. Let $C_{k+1}=W_1 \setminus V_1$. $C_{k+2}=W_2$ and $C_{k+3}=W_3$. Then, it should be clear from the construction that $C_1, C_2, \dots, C_{k+3}$ is a $(k+3)$-coloring of S. Moreover $C_k \cup C_{k+1}=C'$ is also a proper independent set which means that $C_1, C_2, \dots, C_{k-1}, C', C_{k+2}, C_{k+3}$ is a $(k+2)$-coloring of S as well. ▮

Corollary 3.3. $\chi(P) \in [\chi(S)-2, \chi(S)]$. ▮

Having established that one can embed a partial triple system in such a way that the chromatic number of the resulting Steiner triple system is at most slightly larger than that of the initial partial system, then the next point to consider is the efficiency of the embedding algorithm. In this regard there are two basic steps in the above construction; the first is Cruse's construction of an idempotent commutative quasigroup from the partial quasigroup $(V, *)$ and the second step is the actual construction of the triple system. Clearly the second step is quite simple and efficiently computable.

The question becomes then, how efficient in Cruse's embedding of the partial idempotent commutative quasigroup. Since Cruse's proof was constructive, one can define an algorithm based upon it. From a brief study of the proof, the algorithm is obviously polynomial. If one is familiar with Cruse's proof, then it is clear that the construction is based on solving a weighted bipartite matching problem and hence the algorithm will be polynomial. If one is not familiar with that proof, anything short of reproducing it here will be ineffective. Hence, we leave the verification of the following lemma to the interested reader.

Lemma 3.4. *There is a polynomial time algorithm for embedding a partial $S(2, 3, n)$ into a Steiner triple system $S(2, 3, 6n+3)$.* ∎

At this point it should be clear that if there was an efficient algorithm for finding the chromatic number of a Steiner triple system, then there would be an efficient approximation algorithm for the chromatic number of a simple 3-graph.

4. NP-Completeness Results for Partial Triple Systems

To start with, we shall present a polynomial time algorithm, which given a regular graph G of degree four constructs a partial triple system P_G such that,

$$\chi(P_G) = 4\chi(G).$$

Since the problem of recognizing 3-colorable graphs is NP-complete, even when we restrict ourselves to regular graphs of degree four (cf. [8]), a polynomial-time algorithm for finding the chromatic number of a Steiner triple system would imply that P=NP (see Lemma 3.4, Corollary 3.3).

Our algorithm for constructing P_G proceeds in three steps. The first step is to construct a digraph H_G from a given four regular graph G such that $\chi(H_G)=4\chi(G)$ and furthermore the out-degree of any vertex does not exceed 19.

Let $G^*=(V, E)$ be a given acyclic digraph of maximal outdegree $k-1$. Denote by $G^* \otimes G^*$ the following digraph:

$$V(G^* \otimes G^*) = V \cup ([V]^k \times V)$$

$$E(G^* \otimes G^*) = E \cup E_1 \cup E_2$$

where

$$E_1 = \{((b, y), x) | x, y \in V, b \in [V]^k \text{ and } x \in b\}$$

$$E_2 = \{(b, y), (b, z) | b \in [V]^k \text{ and } (y, z) \in E\}.$$

Clearly, the maximal out-degree of $G^* \otimes G^*$ is $2k-1$ and moreover

$$\chi(G^* \otimes G^*) = 2\chi(G^*).$$

(This follows from the elementary fact that an acyclic digraph with maximal out-degree $k-1$ can be k colored.)

Now, let G be a given regular graph of degree 4. Let G^* be any acyclic orientation and set

$$H_G = (G^* \otimes G^*) \otimes (G^* \otimes G^*).$$

It is not hard to see that $\chi(H_G)=4\chi(G)$ and moreover the maximal out-degree of H_G is 19.

Step II involves partitioning the arcs of a digraph $H=(W, F)$ with a maximal out-degree of 19. Set $H^0=H$. Suppose H^i, $0 \leq i < 19$ has been constructed. For each vertex $w \in W$, choose an arc (w', w) (if possible). Denote this set of arcs by F_{i+1} and delete this from H^i. Denote the new graph by H^{i+1}. Clearly $F = \bigcup_{i=1}^{19} F_i$ is a partition and moreover the out-degree of any vertex of (W, F_i) $i=1, 2, \ldots, 19$ is at most one.

In the third step we shall construct a partial triple system P_G such that

$$\chi(P_G) = \chi(H_G) = 4\chi(G).$$

To do this we shall need the following lemma; its proof is postponed until later.

Lemma 4.1. *For all positive integers p, q there exist two disjoint sets A, B with $|A| = |B| = N(p, q)$ and a partial triple system S on $A \cup B$ such that*

(i) $|b \cap A| = 2$, $|b \cap B| = 1$ *for every* $b \in S$.

(ii) *There exists a partition* $S = S_1 \cup S_2 \cup \ldots \cup S_q$ *such that for every* $A' \subset A$, $B' \subset B$,

$$|A'| \geqq \frac{|A|}{p}, \quad |B'| \geqq \frac{|B|}{p}$$

and for every i, $1 \leqq i < q$ *there exists* $b \in S_i$ *where* $b \subset A' \cup B'$.

Now set $p=20$, $q=19$ and $N(20, 19)=N$. Let $(A \cup B, S)$ be the above triple system with these parameters. Let $H_G=(W, F)$. For every $(w', w) \in F$ consider a 1—1 mapping $\varphi_{w',w}\colon A \cup B \to \{w', w\} \times \{1, 2, \ldots, N\}$ such

$$\varphi_{w,w'}[A] = \{w\} \times \{1, 2, \ldots, N\}$$

$$\varphi_{w,w'}[B] = \{w'\} \times \{1, 2, \ldots, N\}.$$

Let $P_G=(X, \tau)$ be a partial triple system with

$$X = W \times \{1, 2, \ldots, N\}$$

$$\tau = \bigcup_{l=1}^{19} \{\varphi_{w',w}[b]\colon (w', w) \in F_l \quad \text{and} \quad b \in S_l\}.$$

To show that (x, τ) is a simple 3-graph or partial triple system it suffices to note that if two triples have a pair in common that pair must belong to a set $\{w\} \times \{1, 2, \ldots, N\}$ the non-existence of such a pair follows from the fact that w is the initial point of at most one arc of F_j, $1 \leqq j \leqq 19$.

Now we prove that $\chi(P_G)=\chi(H_G)$. Let $C_1 \cup C_2 \cup \ldots \cup C_r$ be a coloring of H_G. For $x=(w, i) \in X$ put $x \in C_j^*$ if and only if $w \in C_j$. Clearly $C_1^* \cup C_2^* \cup \ldots \cup C_r^*$ is an r-coloring of P_G and thus

$$\chi(P_G) \leqq \chi(H_G).$$

Let $r=\chi(P_G)$ and let $C_1^* \cup C_2^* \cup \ldots \cup C_r^*$ be an r-coloring of P. (Note: $r \leqq \chi(H_G) \leqq 20$). For every $w \in W$ choose a color C_i^* such that $|C_i^* \cap (\{w\} \times \{1, 2, \ldots, N\})| \geqq N/20$.Define $\psi(w)=i$. According to lemma 4.1 $\psi(w) \neq \psi(w')$ for $(w', w) \in F$ and thus ψ is a coloring of $H_G=(W, F)$—hence $\chi(H_G) \leqq r = \chi(P_G)$.

5. Further lemmas

In the remaining section, we prove Lemma 4.1—it suffices to show the following modification.

Lemma 5.1. *For all positive integers p, q there exist disjoint sets A, B with $|A|=|B|=N(p, q)$ and a simple $4q$-hypergraph S^* on $A\cup B$ such that,*

(i) $|T\cap A|=|T\cap B|=2q$ *for every* $T\in S^*$;

(ii) *for every* $A'\subset A, B'\subset B, A'\geqq\frac{|A|}{p}$, $|B'|\geqq\frac{|B|}{p}$ there exists $T\subset A'\cup B'$, $T\in S^*$.

It remains to prove Lemma 5.1.

Our method will be similar to that of Erdős, Lovász [3]. We shall construct our hypergraph $(A\cup B, S^*)$ inductively. Set

$$c = \ulcorner (10p)^{4q}\log 4\urcorner.$$

$$N(p, q) = N \text{ where } \left(8-\frac{4}{q}\right)N \geqq 20\,qp \quad \text{and}$$

$$N \geqq 16eq^2p^2(4q)^{4q} \text{ hold.}$$

Moreover set, $M=cN$ and $d=20pqc$.

Assume that A, B are disjoint sets of cardinality N. Choose T_1 such that $|T_1\cap A|=|T_1\cap B|=2q$. Suppose that $T_1, T_2, \ldots, T_r$ $(r<M)$ have been constructed so that

1) $|T_i\cap T_j|\leqq 1$ for any $1\leqq i, j\leqq r$, $i\neq j$.

2) No point of $A\cup B$ is contained in more than d of $T_1, T_2, \ldots, T_r$.

3) $|T_i\cap A|=|T_i\cap B|=2q$ for all $1\leqq i\leqq r$.

Let $D_j=A_j\cup B_j$, $1\leqq j\leqq X_r$, be all $(2N/p)$-element subsets of $A\cup B$ such that $|A_j|=|B_j|=N/p$, $A_j\subseteq A$, $B_j\subseteq B$ and $T_i\not\subset D_j$ for any choice of i and j, $1\leqq i\leqq r$, $1\leqq j\leqq X_r$. If there is no such D_j then we are finished. (Our aim is to prove this for some $r_0\leqq M$.) Suppose therefore $X_r\geqq 1$. For every D_j, $1\leqq j\leqq X_r$, we shall estimate the number of $4q$-triples T such that

(a) $T\subset D_j$

(b) $|T\cap A|=|T\cap B|=2q$

(c) $|T\cap T_i|\leqq 1$ for every i, $1\leqq i\leqq r$

(d) T doesn't contain a point which is contained in d of $T_1, T_2, \ldots, T_r$.

Let t_j be the number of points of D_j with degree d (with respect to $T_1, T_2, \ldots$ $\ldots, T_r$). Then, $t_jd\leqq 4qr\leqq 4qM$ and hence $t_j\leqq\frac{4qM}{d}$. Thus the number of points of D_j which are contained in less than d of the $4q$-tuples $T_1, T_2, \ldots, T_r$ is at least,

$$\frac{2N}{p}-t_j \geqq \frac{2N}{p}-\frac{4qM}{d} \geqq \frac{1.8\,N}{P}$$

for every j, $1\leqq j\leqq X_r$ (as $d=20qpc$ and $M=cN$). Hence both A_j and B_j contain at least $0.8\frac{N}{p}$ points with degree less than d. On the other hand there are at most $r\binom{4q}{2}$

distinct pairs of points of D_j that are contained in one of $T_1, T_2, \dots, T_r$. Hence there are at most

$$r\binom{4q}{2}\cdot \operatorname{Max}\left\{\binom{\frac{N}{P}-1}{2q-1}^2, \binom{\frac{N}{P}-2}{2q-2}\binom{\frac{N}{P}}{2q}\right\} < 8rq^2\left(\frac{N}{P}\right)^{4q-2}$$

$4q$-tuples in D_j satisfying (b) but failing to satisfy (c). Hence the total number of $4q$-tuples satisfying (a), (b), (c) and (d) is

$$\binom{0.8\frac{N}{p}}{2q}^2 - 8Mq^2\left(\frac{N}{p}\right)^{4q-2} \geqq \left(\frac{.8\frac{N}{p}-2q}{2q}\right)^{4q} - 8Mq^2\left(\frac{N}{p}\right)^{4q-2}$$

$$\geqq \left(\frac{N}{4pq}\right)^{4q} - 8Mq^2\left(\frac{N}{p}\right)^{4q-2} \geqq \left(\frac{N}{p}\right)^{4q}\left[\left(\frac{1}{4q}\right)^{4q} - \frac{8Mq^2}{\left(\frac{N}{p}\right)^2}\right]$$

$$= \left(\frac{N}{P}\right)^{4q}\left[\left(\frac{1}{4q}\right)^{4q} - \frac{8cq^2p^2}{N}\right] \geqq \frac{1}{2}\left(\frac{N}{4pq}\right)^{4q}.$$

(Here we used the facts $N \geqq 20qp$ and $N \geqq 16c^2p^2q^2(4q)^{4q}$.)

As the total number of possible $4q$-tuples (i.e. such that $|T\cap A| = |T\cap B| = 2q$) is $\binom{N}{2q}^2$, there is one, say T_{r+1}, which satisfies (b)—(d) and is contained in at least

$$X_r\cdot\frac{\frac{1}{2}\left(\frac{N}{4pq}\right)}{\binom{N}{2q}^2} > \frac{X_r}{2}\left(\frac{1}{2ep}\right)^{4q} > \frac{X_r}{(10p)^{4q}}$$

of the subsets D_j. Hence there are at most $X_{r+1} \leqq X_r\left(1-\frac{1}{(10p)^{4q}}\right)$ of the D_j, $1 \leqq j \leqq X_r$ containing none of $T_1, \dots, T_{r+1}$. As $X_0 = \binom{N}{N/p}^2$ we get that,

$$x_M \leqq x_0\left(1-\frac{1}{(10p)^{4q}}\right)^M < 2^{2N}e^{M\log(1-1/(10p)^{4q})} < e^{N\log 4 - 1/(10p)^{4q}} \leqq 1,$$

and hence after $r_0 \leqq M$ steps we get a hypergraph with the required properties.

6. Conclusions

The construction presented in the previous sections, guarantees that if there was a polynomial time algorithm for determining c such that $\chi(P)\in(c-4, c]$ for any simple 3-graph P then one could complete $\chi(G)$, for any 4 regular graph G in polynomial time. This implies that P=NP. Since determining $\chi(S)$ for a Steiner triple system, S, would give such an algorithm, we conclude that deciding the chromatic number of a Steiner triple system is NP-hard.

Corollary 6.1. *Deciding 14-colorability of a Steiner triple system is NP-complete.*

Proof. Deciding 3-colorability for a 4-regular graph G is NP-complete. Construct P_G from G. Since $\chi(P_G)=4\chi(G)$, then if $\chi(G)\geqq 4$, $\chi(P_G)\geqq 16$. Embedding P_G into a triple system S means that $\chi(S)\geqq 16$. However if $\chi(G)\leqq 3$ then $\chi(P_G)\leqq 12$ and $\chi(S)\leqq 14$. ▮

Though we did not need it in this paper, the following is an easy and interesting corollary:

Corollary 6.2. *Deciding 2-colorability of simple k-graphs is NP-complete.*

Proof. Deciding 2-colorability for k-graphs is NP-complete (Lovász [13]). We transform the k-graph into a simple k-graph. Whenever the pair $\{x, y\}$ occurs in more than one subset replace each additional occurrence by a new pair of elements $\{x', y'\}$ and use a 2-chromatic "animal" from Lemma 2.1 to identify x and x' (and y and y') so that any 2-coloring of the animal must assign the same colors to x and x' (and y and y'). We repeat this process until every pair appears at most once. The transformation is polynomial and any 2-coloring of the transformed hypergraph induces a 2-coloring of the original- and vice versa. ▮

The above arguments can be used to give an alternate proof of Theorem 2.4.

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